Homological algebra exercise sheet Week 12

1. The goal of this exercise is to complete the proof showing that $K(\mathcal{A})$ is triangulated. Recall that to prove (TR4) one needs to show the following diagram commutes and that $(\mathrm{cone}(u), \mathrm{cone}(vu), \mathrm{cone}(v))$ is an exact triangle:

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, equal] & B \arrow[r] \arrow[d, "v"] & \mathrm{cone}(u) \arrow[r] \arrow[d] & A[-1] \arrow[d, equal] \\
A \arrow[r, "vu"] & C \arrow[r] \arrow[d] & \mathrm{cone}(vu) \arrow[r] \arrow[d] & A[-1] \\
& \mathrm{cone}(v) \arrow[r, equal] \arrow[d] & \mathrm{cone}(v) \arrow[d] & \\
& B[-1] \arrow[r] & \mathrm{cone}(u[-1]). &
\end{tikzcd}$$

The only part remaining to show that $(\mathrm{cone}(u), \mathrm{cone}(vu), \mathrm{cone}(v))$ is indeed a triangle. By definition of triangles in $\mathcal{K}(\mathcal{A})$, it suffices to show the commutativity of the following diagram in $\mathcal{K}(\mathcal{A})$, plus that γ is a homotopy equivalent in $\mathbf{Ch}((\mathcal{A}))$. We have defined the maps $v \oplus \mathrm{id}, \mathrm{id} \oplus u, \pi_B, \gamma$ in the lecture and one may easily verify every required condition except for the following:

   (a) $\gamma \circ (\mathrm{id} \oplus u)$ is homotopic to ι.

   (b) γ defines a homotopy equivalence between the two chain complexes.

$$\begin{tikzcd}
B \oplus A[-1] \arrow[r, "v\oplus\mathrm{id}"] \arrow[d, equal] & C \oplus A[-1] \arrow[r, "\mathrm{id}\oplus u"] \arrow[d, equal] & C \oplus B[-1] \arrow[r, "\pi_B"] \arrow[d, "\gamma"] & B[-1] \oplus A[-2] \arrow[d, equal] \\
B \oplus A[-1] \arrow[r, "v\oplus\mathrm{id}"] & C \oplus A[-1] \arrow[r, "\iota"] & (C \oplus A[-1]) \oplus (B[-1] \oplus A[-2]) \arrow[r] & B[-1] \oplus A[-2]
\end{tikzcd}$$

As mentioned one may construct γ as the natural inclusion and the homotopy inverse of γ as

$$g : (c, a', b', a'') \to (c, u(a') + b').$$

By representing the morphisms as matrices, find the suitable chain homotopies and show (a) and (b).

Hint: You may need the matrix corresponding to the boundary map of $(C \oplus A[-1]) \oplus (B[-1] \oplus A[-2])$, *which is given by* $\begin{pmatrix} \partial & -vu & -v & 0 \\ 0 & -\partial & 0 & \mathrm{id} \\ 0 & 0 & -\partial & u \\ 0 & 0 & 0 & \partial \end{pmatrix}$.
The other matrices you need should be rather trivial.

2. Let S be a locally small multiplicative system of morphisms in a category $\mathcal{C}$. Let $q : \mathcal{C} \to S^{-1}\mathcal{C}$ be the localization constructed in class.

   (a) Let Z be a zero object in $\mathcal{C}$. Show that $q(Z)$ is a zero object. Deduce that for every X in $\mathcal{C}$ we have
   $$q(X) \cong 0 \iff S \text{ contains the zero map } X \xrightarrow{0} X.$$

   (b) Assume that the product $X \times Y$ exists in $\mathcal{C}$. Show that
   $$q(X \times Y) \cong q(X) \times q(Y)$$
   in $S^{-1}\mathcal{C}$.

   (c) Assume now that C is an additive category. Show that $S^{-1}\mathcal{C}$ is also additive and that q is an additive functor.

3. Let R be a commutative ring and $S \subseteq R$ be multiplicatively closed subset. Let **mod-**R be the category of R-modules. Let Σ be the collection of all morphisms $A \to B$ in **mod-**R such that the induced morphism on localizations $S^{-1}A \to S^{-1}B$ is an isomorphism. Show that **mod-**$S^{-1}R$ is a localizing subcategory of **mod-**R and
$$\textbf{mod-}S^{-1}R \cong \Sigma^{-1}\textbf{mod-}R.$$

4. Let $\mathcal{A}$ be an abelian category. An abelian subcategory $\mathcal{B}$ of $\mathcal{A}$ is called a *Serre subcategory* if it is closed under sub-objects, quotients, and extensions. In other words, this means that for every short exact sequence
$$0 \to A' \to A \to A'' \to 0$$
in $\mathcal{A}$, the object A is in $\mathcal{B}$ if and only if both A' and A'' are in $\mathcal{B}$.

   Suppose that $\mathcal{B}$ is a Serre subcategory of $\mathcal{A}$ and let Σ be the family of all morphisms f in $\mathcal{A}$ with $\ker(f)$ and $\mathrm{coker}(f)$ in $\mathcal{B}$.

   (a) Show that Σ is a multiplicative system in $\mathcal{A}$. We write $\mathcal{A}/\mathcal{B}$ for the localization $\Sigma^{-1}\mathcal{A}$ (provided that it exists).

   (b) Show that $q(X) \cong 0$ in $\mathcal{A}/\mathcal{B}$ if and only if X is in $\mathcal{B}$.

   (c) Assume that $\mathcal{B}$ is a small category, and show that Σ is locally small. This is one case in which $\mathcal{A}/\mathcal{B} = \Sigma^{-1}\mathcal{A}$ exists.

(d) Show that $\mathcal{A}/\mathcal{B}$ is an abelian category and that $q : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ is an exact functor.

(e) Let S be a multiplicative subset of a commutative ring R (in the usual sense), and let $\mathbf{mod}_S R$ be the full subcategory of R-modules A such that $S^{-1}A \cong 0$. Show that $\mathbf{mod}_S R$ is a Serre subcategory of $\mathbf{mod}$-R. Conclude that $\mathbf{mod}\text{-}S^{-1}R \cong \mathbf{mod}\text{-}R/\mathbf{mod}_S R$.

5. The goal of this exercise is to show that $\mathbf{K}(\mathrm{Ab})$ is not abelian. Let $\mathcal{K}$ be a triangulated category.

(a) Let
$$A \to B \to C \to TA$$
and
$$A' \to B' \to C' \to TA'$$
be exact triangles in $\mathcal{K}$. Show that there is an exact triangle
$$A \oplus A' \to B \oplus B' \to C \oplus C' \to T(A \oplus A').$$

(b) Let
$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA$$
be a triangle in $\mathcal{K}$ and assume that w is zero. Deduce from (a) that
$$B \cong A \oplus C.$$
Hint: Show that the triangles
$$A \xrightarrow{\mathrm{id}} A \to 0 \to TA$$
and
$$0 \to C \xrightarrow{\mathrm{id}} C \to 0$$
are exact using (TR4)*, or any other way. Conclude using* (TR3).

(c) Let $f : A \to B$ be a monic in $\mathcal{K}$. Deduce from (b) that
$$B \cong A \oplus C$$
for some C in $\mathcal{K}$.

(d) We now restrict to $\mathcal{K} = \mathbf{K}(\mathrm{Ab})$. Show that $\mathbf{K}(\mathrm{Ab})$ is not abelian by considering the map
$$\mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}.$$